

Radiation from charged particles on eccentric orbits in a dipolar magnetic field around a Schwarzschild black hole

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Abstract We obtain an approximate solution for the motion of a charged particle around a Schwarzschild black hole immersed in a weak dipolar magnetic field. We focus on eccentric bound orbits in the equatorial plane of the Schwarzschild black hole and derive an analytic expression for the spectral distribution of the electromagnetic emission from a charged particle on such an orbit. Two sets of harmonic contributions appear, with specific frequency spacing. The expression can be written in compact form, if it is truncated up to the lowest order harmonic contributions.

1 Introduction

The origin of astrophysical magnetic fields on galactic and extragalactic scales remains one of the open questions of modern astrophysics. Microgauss magnetic fields have been observed in elliptical and spiral galaxies, in the intracluster medium, and in damped Lyman alpha systems over cosmological distances (see e.g. [1] for a review). In the past, several investigations of bremsstrahlung radiation were carried out in the case of a Schwarzschild black hole embedded in a large-scale homogeneous asymptotically uniform external magnetic field ([2–4]). The total radiated power was obtained through a generalized Larmor formula based on arguments about covariance under Lorentz transformations ([5, 6]). In a broad sense, bremsstrahlung [7] is the radiation emitted by an accelerated charged particle. The angular distribution of the radiation produced from an orbiting charged particle in the presence of a uniform magnetic field has been discussed extensively with the aid of the Lienard-Wiechert potentials at large distances in [6, 8, 9]. Along the way, several interesting features of the dynamics of charged particles in the presence of both the black hole gravitational field and the uniform magnetic field were investigated. In [10] and references therein,

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the authors explored two different types of orbits, with and without curls. This analysis was generalized by [11] in the case of a Schwarzschild black hole embedded in a dipole magnetic field. An interesting result obtained in [12] was the possibility of trapping relativistic charged particles in a potential well inside the position of the innermost stable circular orbit. In the present work, we investigate the eccentric orbits of charged particles around a Schwarzschild black hole embedded in a dipole field and obtain their electromagnetic radiation spectrum, assuming slow motions and weak fields. Our paper is organized as follows: In Section 2 we present the mathematical formalism. In Section 3, we investigate eccentric bound orbits in the equatorial plane and in Section 4, we discuss the angular distribution of the electromagnetic radiation from those particles. We close with a discussion in Section 5.

2 Equations of motion in Schwarzschild space-time

We follow the $3+1$ formalism of [5] and [13] for a dipole magnetic field in a Schwarzschild background and work in geometric units with $c = G = 1$ (in some equations the speed of light c is kept on purpose). Greek spacetime indices take values from 0 to 3 and Latin indices from 1 to 3. In a (t, r, θ, ϕ) coordinate system the Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{r_S}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_S}{r}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (1)$$

where $r_S = 2M$ is the Schwarzschild radius of the black hole. The components of the covariant four-velocity are $u^\mu = (u^0, u^i)$, with $u_\mu u^\mu = -1$. The equations of motion of a charged test particle in the presence of a dipole magnetic field can be derived from the Lagrangian

$$2\mathcal{L} = g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + \frac{q}{m} \frac{dx^\mu}{d\tau} A_\mu, \quad (2)$$

where τ is the proper time and m and q are the particle's mass and electric charge respectively. The Lagrangian now reads

$$2\mathcal{L} = -\left(1 - \frac{r_S}{r}\right) \dot{t}^2 + \frac{\dot{r}^2}{1 - \frac{r_S}{r}} + r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) + \frac{q}{m} \frac{dx^\mu}{d\tau} A_\mu. \quad (3)$$

where a dot denotes differentiation with respect to τ .

The motion of a charged particle around a Schwarzschild black hole immersed in a dipole magnetic field [11], [12] is characterized by two conserved quantities, the energy $E > 0$ and the generalized angular momentum L , given by the expressions

$$\dot{t} = E \left(1 - \frac{r_S}{r}\right)^{-1}, \quad (4)$$

$$\dot{\phi} = \frac{L}{r^2 \sin^2 \theta} - \frac{q}{m} \frac{A_\phi}{r^2 \sin^2 \theta}, \quad (5)$$

$$\dot{\theta} = \frac{p_\theta}{r^2}. \quad (6)$$

We assume a dipole electromagnetic four-potential of the form $A_\mu = (0, 0, 0, A_\phi)$, where

$$A_\phi = \frac{3\mu_b}{r_S^3} r^2 \sin^2 \theta W(r), \quad (7)$$

and

$$W(r) \equiv \ln \left(1 - \frac{r_S}{r} \right) + \frac{r_S}{r} + \frac{r_S^2}{2r^2}. \quad (8)$$

Above, μ_b is the magnetic dipole moment for an observer at infinity. Thus, for the metric (1) the normalization of the 4-velocity yields

$$\dot{r}^2 = E^2 - \left(1 - \frac{r_S}{r} \right) \left[1 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \left(\frac{L}{r^2 \sin^2 \theta} - \frac{3q\mu_b}{mr_S^3} W(r) \right)^2 \right]. \quad (9)$$

We make the following changes in the above equation. First, we change the τ -derivatives to ϕ -derivatives, through the relation $\dot{r} = (dr/d\phi)\dot{\phi}$; second, we assume that the particle motion is confined in the equatorial plane ($\theta = \pi/2$); finally, we assume that $r \gg r_S$, thus $W(r) \approx -r_S^3/3r^3$, and we arrive at the following equation of motion, which describes the geometry of charged particle orbits in the invariant equatorial plane

$$\left(\frac{du}{d\phi} \right)^2 = \frac{1}{(1 + \Omega u)^2} \left[\frac{E^2 - 1}{L^2} + \frac{r_S}{L^2} u - u^2 + (r_S - 2\Omega) u^3 + (2\Omega r_S - \Omega^2) u^4 + \Omega^2 r_S u^5 \right], \quad (10)$$

where $u = 1/r$ and

$$\Omega \equiv \frac{q\mu_b}{mL}. \quad (11)$$

The parameter Ω can be negative or positive, depending on the sign of the charge q and on the azimuthal generalized angular momentum L . In what follows, we will restrict ourselves to positive values of Ω .

3 Bound orbits

The motion of charged particles in a dipolar magnetic field around a static black hole was studied in [11, 12] and [15] (see also the review in [2]). A synchrotron mechanism was considered for circular orbits of charged particles in the presence of a uniform magnetic field. Here, we focus only on eccentric bound orbits of charged particles of the form $r = l/(1 + e \cos \xi)$, where e is the eccentricity of the orbit with $0 < e < 1$, l is its latus rectum and $\xi \in [0, 2\pi]$ in the equatorial plane of a Schwarzschild black hole, immersed in a dipole magnetic field, when $r/r_S \gg 1$. We seek approximate, analytic solutions, by keeping only orders up to $O(u^3)$ in the series appearing in the nominator of (10). With theses assumptions, we find values of E , L and Ω such that Eq. (10) admits bound orbits. More details on this method can be found in [16–19].

The geometry of the orbits is determined by the real roots of the equation $f_1(u) = 0$ where

$$f_1(u) = -\frac{1-E^2}{L^2} + \frac{r_S}{L^2}u - u^2 + (r_S - 2\Omega)u^3. \quad (12)$$

(when $1 + \Omega u \neq 0$). The orbital eccentricity e and latus rectum $l = a(1 - e^2)$ (with a being the major axis of the ellipse) satisfy

$$\frac{r_S}{L^2} = \frac{1}{l^2} [2l - (3 + e^2)(r_S - 2\Omega)], \quad (13)$$

$$\frac{(1-E^2)}{L^2} = \frac{1}{l^3} [l - 2(r_S - 2\Omega)](1 - e^2). \quad (14)$$

In this case, the equation $f_1(u) = 0$ admits three real roots which are

$$u_1 = \frac{1-e}{l}, \quad u_2 = \frac{1+e}{l}, \quad u_3 = \frac{1}{r_S - 2\Omega} - \frac{2}{l}. \quad (15)$$

From (13) we deduce that

$$\mu \equiv \frac{M}{l} \leq \frac{1}{3+e^2} + \frac{\Omega}{l}, \quad (16)$$

and assuming that $u_3 \geq u_2$ we arrive at another interesting inequality

$$r_S - 2\Omega \leq \frac{l}{3+e}. \quad (17)$$

Substituting (13), (14) and

$$u = \frac{1}{l}(1 + e \cos \xi), \quad (18)$$

into (10), we obtain

$$\left(\frac{d\xi}{d\phi} \right)^2 = \frac{1}{[1 + (\Omega/l)(1 + e \cos \xi)]^2} \left[1 - (3 + e \cos \xi) \frac{r_S - 2\Omega}{l} \right], \quad (19)$$

which reduces to

$$\left[1 + \frac{\Omega}{l}(1 - e) \right] \frac{d\xi}{\sqrt{1 - k^2 \cos^2(\xi/2)}} - \left(\frac{2\Omega e}{l} \right) \frac{\cos^2(\xi/2) d\xi}{\sqrt{1 - k^2 \cos^2(\xi/2)}} = \pm \sqrt{\lambda} d\phi, \quad (20)$$

with $\lambda \equiv 1 - (3 - e) \frac{r_S - 2\Omega}{l} > 0$ and

$$k^2 = \frac{2e(r_S - 2\Omega)}{l - (3 - e)(r_S - 2\Omega)} < 1. \quad (21)$$

Integrating (20) we find

$$2[l + \Omega(1 - e)] F[\cos(\xi/2), k] - 4\Omega e \{F[\cos(\xi/2), k] - E[\cos(\xi/2), k]\} = \pm \sqrt{\lambda} l \phi, \quad (22)$$

where $F[\cos(\xi/2), k]$ and $E[\cos(\xi/2), k]$ are the elliptic integrals of first and second kind, respectively, with modulus $k < 1$.

Expanding the elliptic integrals in terms of the small modulus k and keeping only zero order terms in $k < 1$ (see [20–22]), we obtain

$$\xi - \frac{\Omega e}{l + \Omega(1 - 2e)} \sin \xi = \pi \pm \chi \phi. \quad (23)$$

where $\frac{\Omega e}{l + \Omega(1 - 2e)} < 1$ and $\chi = \frac{l\sqrt{\lambda}}{l + \Omega(1 - 2e)}$.

From (23) we compute $\cos \xi$, which appears in (18) and obtain

$$\cos \xi = -\cos(\chi \phi) \mp \frac{\Omega e}{l + \Omega(1 - 2e)} \sin^2(\chi \phi). \quad (24)$$

Substitution of (24) into (18) yields

$$u = \frac{1}{l} \left[1 - e \cos(\chi \phi) \mp \frac{\Omega e^2}{l + \Omega(1 - 2e)} \sin^2(\chi \phi) \right]. \quad (25)$$

To linear order in e , the last equation becomes

$$u = \frac{1}{l} [1 - e \cos(\chi \phi)], \quad (26)$$

where the origin of ϕ may be determined by Eq.(23) setting $\xi = 0$.

The approximate solution may now be completed by direct integration of the equations

$$\begin{aligned} \frac{d\phi}{d\tau} &= Lu^2(1 + \Omega u), \\ \frac{dt}{d\tau} &= E(1 - r_s u)^{-1}. \end{aligned} \quad (27)$$

From (26) and (27) we obtain

$$\frac{d(\chi \phi)}{[1 - e \cos(\chi \phi)]^2 [\rho_1 + \rho_2 \cos(\chi \phi)]} = \frac{\chi L}{El^2} dt, \quad (28)$$

where we have neglected the $O(e^2)$ term $2e^2(\frac{\mu\Omega}{l})$ and set $\rho_1 = (1 - 2\mu)(1 + \frac{\Omega}{l})$ and $\rho_2 = 2\mu - \frac{\Omega}{l}(1 - 4\mu)$.

Integrating (28), we find

$$\begin{aligned} Nt \equiv & B_1 \arctan \left(\sqrt{\frac{1+e}{1-e}} \tan \frac{(\chi \phi)}{2} \right) + B_2 \arctan \left(\sqrt{\frac{\rho_1 + e\rho_2}{\rho_1 - e\rho_2}} \tan \frac{(\chi \phi)}{2} \right) \\ & + B_3 \frac{\sin(\chi \phi)}{2(1 - e \cos(\chi \phi))}, \end{aligned} \quad (29)$$

where

$$\begin{aligned} B_1 &= \frac{2[(\rho_1 + \rho_2) + \rho_2(1 - e^2)]}{(1 - e^2)^{3/2}(\rho_1 + \rho_2)^2}, \\ B_2 &= \frac{2\rho_2^2}{(\rho_1 + \rho_2)^2 \sqrt{\rho_1^2 - e^2 \rho_2^2}}, \\ B_3 &= \frac{2e}{(1 - e^2)(\rho_1 + \rho_2)}, \end{aligned} \quad (30)$$

and

$$N \equiv \frac{\chi L}{2El^2} = \frac{L}{2El^2} \frac{\sqrt{1 - (3 - e)\frac{r_s - 2\Omega}{l}}}{1 + \frac{\Omega}{l}(1 - 2e)}, \quad (31)$$

has dimensions of frequency. In Appendix B, we compute the right-hand of (31) in terms of e , μ and $\Omega \neq 0$ with the aid of (13), (14) and find a rather lengthy expression. However, for $\Omega = 0$, we compute the ratio L/E using (13) and recover the Newtonian frequency of a Keplerian orbit with the same e and l .

Note that, when both the eccentricity $e \in (0, 1)$ and $\mu = M/l = r_s/(2l)$ are small, the coefficients B_1 , B_2 and B_3 can be approximated as

$$B_1 = 2 + O(e) \sim 2, \quad B_2 = O(\mu^2), \quad B_3 = O(e). \quad (32)$$

Thus, (29) takes the approximate form

$$\tan\left(\frac{\chi\phi}{2}\right) \simeq \sqrt{\frac{1-e}{1+e}} \tan(Nt), \quad (33)$$

since the second and the third term are very small in comparison to the first one multiplied by B_1 . Furthermore, using (33) we find

$$\begin{aligned} \cos(\chi\phi) &= \frac{\cos(2Nt) - e}{1 - e\cos(2Nt)}, \\ \sin(\chi\phi) &= \sqrt{1 - e^2} \left(\frac{\sin(2Nt)}{1 - e\cos(2Nt)} \right). \end{aligned} \quad (34)$$

Finally, because of our previous approximations in (32), Eq. (26) yields

$$r = \frac{l[1 - e\cos(2Nt)]}{(1 + e^2) - 2e\cos(2Nt)}. \quad (35)$$

4 Spectral distribution of radiation

We are next interested in examining the electromagnetic emission of a particle with charge q that is following an eccentric bound orbit, such as the one discussed in Section III. Our aim is to obtain analytic expressions for the distribution of energy as a function of angle and radiation frequency (see e.g. [6, 8, 9]), valid under the assumptions made in Section III. We make use of the well-known result [6, 8, 9]

$$\frac{d^2 \mathcal{E}}{d\omega' d\Omega_s} = \frac{q^2 \omega'^2}{4\pi^2} \left| \int_{-\infty}^{\infty} \mathbf{n} \times (\mathbf{n} \times \mathbf{v}) e^{i\omega'(t - \mathbf{n} \cdot \mathbf{r}(t))} dt \right|^2, \quad (36)$$

where \mathcal{E} is the total energy radiated per unit frequency ω' per unit solid angle Ω_s [8] in which we replace the internal and external vector products with general-relativistic expressions (derived in the $3+1$ formalism [12, 13]). For our further derivations, it is necessary to know the spatial velocity \mathbf{v} and position $\mathbf{r}(t)$ over a small arc of the trajectory whose tangent is pointing towards the observation point. In our case the segment of the trajectory lies in the xy -plane, and the unit vector \mathbf{n} in the direction of radiation lies (without loss of generality) in the yz -plane at an angle λ with the z -axis. In this case, the unit vector \mathbf{n} is

$$\mathbf{n} = \sin \lambda \mathbf{e}_y + \cos \lambda \mathbf{e}_z, \quad (37)$$

where \mathbf{e}_y , and \mathbf{e}_z are the unit vectors along the y and z axes, respectively. In [6, 8] the authors examined the angular distribution of the energy radiated per unit frequency ω' per unit solid angle Ω_s for $\lambda \approx \pi/2$, since appreciable radiation intensity is concentrated mainly in the equatorial plane of the orbit, and the duration of the pulse is short. Here, we find an expression for any angle $\lambda \in (0, 2\pi)$ (but which is valid only under the assumptions mentioned in Section III).

The internal and external vector products in (36) are derived in spherical polar coordinates and become ¹

$$\mathbf{n} \cdot \mathbf{r} = \gamma_{ij} n^i r^j = \frac{r}{\sqrt{1 - r_S/r}} [\sin \lambda \sin \theta \sin \phi + \cos \theta \cos \lambda], \quad (38)$$

and

$$\mathbf{n} \times (\mathbf{n} \times \mathbf{v}) = X \mathbf{e}_r + \frac{Y}{r} \mathbf{e}_\theta + \frac{Z}{r \sin \theta} \mathbf{e}_\phi, \quad (39)$$

where the explicit expressions of X, Y and Z are given in Appendix A. In the equatorial plane ($\theta = \pi/2$), (36) then becomes:

$$\frac{d^2 \mathcal{E}}{d\omega' d\Omega_s} = \frac{q^2 \omega'^2}{4\pi^2} \left| \int_{-\infty}^{\infty} \left[X_0 \mathbf{e}_r + \frac{Y_0}{r} \mathbf{e}_\theta + \frac{Z_0}{r} \mathbf{e}_\phi \right] e^{i\omega' t} \times \exp \left\{ -i\omega' \frac{r \sin \lambda \sin \phi}{\sqrt{(1 - r_S/r)}} \right\} dt \right|^2, \quad (40)$$

¹ The cross vector product is $(\mathbf{n} \times \mathbf{v})_l = \varepsilon_{lkm} n^k v^m$ with $\varepsilon_{123} = \frac{r^2 \sin \theta}{\sqrt{(1 - r_S/r)}}$, while in the dot product γ_{ij} is the spatial part of the metric.

where

$$X_0 \equiv -\frac{1}{\sqrt{(1-r_S/r)}} \left[\cos^2 \lambda \frac{dr}{dt} - \sin^2 \lambda \cos \phi \frac{d(r \cos \phi)}{dt} \right], \quad (41)$$

$$Y_0 \equiv -r \sin \lambda \cos \lambda \frac{d(r \sin \phi)}{dt}, \quad (42)$$

$$Z_0 \equiv -r \left[\sin^2 \lambda \sin \phi \frac{d(r \cos \phi)}{dt} - \cos^2 \lambda \left(r \frac{d\phi}{dt} \right) \right]. \quad (43)$$

Using (67), the exponential function in (40) becomes

$$e^{i\omega'(t-\mathbf{n}\cdot\mathbf{r}(t))} = \sum_{m=-\infty}^{\infty} J_m(Q_1) e^{it(\omega'-2mN)} + \sum_{m=-\infty}^{\infty} J_m(Q_2) e^{it(\omega'-4mN)}, \quad (44)$$

where J_m are the Bessel functions of the first kind, while we set

$$Q_1 = \omega' \sin \lambda \frac{l}{\sqrt{1-2\mu}} \quad \text{and} \quad Q_2 = \omega' \sin \lambda \frac{l(2-5\mu)e}{2(1-2\mu)^{3/2}}. \quad (45)$$

Eq. (40) with the help of (64) and (44) now reads

$$\frac{d^2 \mathcal{E}}{d\omega' d\Omega_s} = \frac{q^2 \omega'^2}{4\pi^2} [(W_1^m)^2 + (W_2^m)^2 + (W_3^m)^2], \quad (46)$$

where

$$W_1^m \equiv \int_{-\infty}^{\infty} e^{i\omega'(t-\mathbf{n}\cdot\mathbf{r}(t))} X_0 dt = 2i\sqrt{\frac{\pi}{2}} l N \left(\sum_{m=-\infty}^{+\infty} \delta(\omega' - 4Nm) W_{1b}^m - \sum_{m=-\infty}^{+\infty} \delta(\omega' - 2Nm) W_{1a}^m \right), \quad (47)$$

$$W_2^m \equiv \int_{-\infty}^{\infty} e^{i\omega'(t-\mathbf{n}\cdot\mathbf{r}(t))} \frac{Y_0}{r} dt = -\sqrt{\frac{\pi}{2}} l N \sin 2\lambda \left(\sum_{-\infty}^{\infty} \delta(\omega' - 2mN) W_{2a}^m + \sum_{-\infty}^{\infty} \delta(\omega' - 4mN) W_{2b}^m \right), \quad (48)$$

$$W_3^m \equiv \int_{-\infty}^{\infty} e^{i\omega'(t-\mathbf{n}\cdot\mathbf{r}(t))} \frac{Z_0}{r} dt = -2\sqrt{\frac{\pi}{2}} l N \left(\sum_{-\infty}^{\infty} \delta(\omega' - 2mN) W_{3a}^m + \sum_{-\infty}^{\infty} \delta(\omega' - 4mN) W_{3b}^m \right). \quad (49)$$

Furthermore, (46) with the aid of (47), (47) and (49) yield

$$\begin{aligned} \frac{d^2 \mathcal{E}}{d\omega' d\Omega_s} &= \frac{T}{4\pi^2} q^2 \omega'^2 l^2 N^2 \\ &\times \left\{ \sum_{m=-\infty}^{+\infty} \delta(\omega' - 2Nm) [(W_{1a}^m)^2 + (W_{2a}^m)^2 + (W_{3a}^m)^2] \right. \\ &\quad \left. + \sum_{m=-\infty}^{+\infty} \delta(\omega' - 4Nm) [(W_{1b}^m)^2 + (W_{2b}^m)^2 + (W_{3b}^m)^2] \right\}. \end{aligned} \quad (50)$$

We have used the approximation that $\delta^2(\omega') = \frac{1}{2\pi/T} \delta(\omega')$ in the limit $T \rightarrow \infty$, where T is the radiation emission time during the orbital motion of the charged particle around the black hole, while we set

$$\begin{aligned} W_{1a}^m &= 2 \cos^2 \lambda (1 - 2\mu)^{1/4} e [J_{m+1}(Q_1) - J_{m-1}(Q_1)] \\ &\quad - \sin^2 \lambda \left\{ \frac{(1 - 2\mu)^2}{2} [J_{m+2}(Q_1) - J_{m-2}(Q_1)] \right. \\ &\quad \quad \left. - 2(1 - 2\mu)^2 e [J_{m+1}(Q_1) - J_{m-1}(Q_1)] \right. \\ &\quad \quad \left. + \frac{(1 - 2\mu)(7 - 6\mu)e}{8} [J_{m+3}(Q_1) - J_{m-3}(Q_1) + J_{m+1}(Q_1) - J_{m-1}(Q_1)] \right\}, \end{aligned} \quad (51)$$

$$\begin{aligned} W_{1b}^m &= \cos^2 \lambda (1 - 2\mu)^{1/4} e [J_{2m+1}(Q_2) - J_{2m-1}(Q_2)] \\ &\quad - \sin^2 \lambda \left\{ \frac{(1 - 2\mu)^2}{2} [J_{2m+2}(Q_2) - J_{2m-2}(Q_2)] \right. \\ &\quad \quad \left. - 2(1 - 2\mu)^2 e [J_{2m+1}(Q_2) - J_{2m-1}(Q_2)] \right. \\ &\quad \quad \left. + \frac{(1 - 2\mu)(7 - 6\mu)e}{8} [J_{2m+3}(Q_2) - J_{2m-3}(Q_2) + J_{2m+1}(Q_2) - J_{2m-1}(Q_2)] \right\}, \end{aligned} \quad (52)$$

$$W_{2a}^m = [J_{m-1}(Q_1) + J_{m+1}(Q_1)] + 2e[J_{m-2}(Q_1) + J_{m+2}(Q_1)], \quad (53)$$

$$W_{2b}^m = [J_{2m-1}(Q_2) + J_{2m+1}(Q_2)] + 2e[J_{2m-2}(Q_2) + J_{2m+2}(Q_2)], \quad (54)$$

$$\begin{aligned} W_{3a}^m &= \sin^2 \lambda \left\{ \frac{1}{2} [2J_m(Q_1) + J_{m-2}(Q_1) - J_{m+2}(Q_1)] \right. \\ &\quad \left. + \frac{5e}{4} [J_{m-1}(Q_1) + J_{m+1}(Q_1) - J_{m-3}(Q_1) - J_{m+3}(Q_1)] \right\} \\ &\quad + 2e \cos^2 \lambda [J_m(Q_1) + J_{m-1}(Q_1) + J_{m+1}(Q_1)], \end{aligned} \quad (55)$$

$$\begin{aligned} W_{3b}^m &= \sin^2 \lambda \left\{ \frac{1}{2} [2J_m(Q_2) + J_{2m-2}(Q_2) - J_{2m+2}(Q_2)] \right. \\ &\quad \left. + \frac{5e}{4} [J_{2m-1}(Q_2) + J_{2m+1}(Q_2) - J_{2m-3}(Q_2) - J_{2m+3}(Q_2)] \right\} \\ &\quad + \frac{e}{2} \cos^2 \lambda [4J_m(Q_2) + J_{2m-1}(Q_2) + J_{2m+1}(Q_2)J_{2m+3}(Q_2) - J_{2m-3}(Q_2)]. \end{aligned} \quad (56)$$

Recall, that the argument Q_2 depends on the geometry of the orbit, since it is proportional to the eccentricity. The analytic expression (50) is our final result.

5 Discussion

We obtained an analytic expression for the spectral distribution of the electromagnetic emission from charged particles in an eccentric, bound equatorial orbit around a Schwarzschild black hole, embedded in a dipolar magnetic field. Our final result (50) includes an infinite number of harmonics. We note that in the second sum, the argument Q_2 in the Bessel functions depends on the eccentricity of the charged particle, but in the first sum, the argument Q_1 does not. It is evident from the analytic expression (50) that the spectrum of the emitted radiation is composed of spectral lines appearing at frequencies intervals $\Delta\omega_1 = 2N$ and $\Delta\omega_2 = 4N$, where N is given by (31).

Our main result (50) can be further integrated over all frequencies ω' and all angles, yielding a distribution in terms of the harmonics m . The relative contribution of individual harmonics will depend on the precise orbit. For small eccentricities, such as those considered here, we expect only the first few harmonics to be important. Under this assumption, one could thus truncate the infinite series in (50), in order to arrive at a more compact analytic expression. We can obtain a rough estimate of the total radiated power for an electron in orbit around a ten solar mass Schwarzschild black hole with $M = 2 \times 10^34$ g, $r_S = 3 \times 10^6$ cm, immersed in a dipole magnetic field typical of intergalactic space of order $B \sim 1\mu\text{G}$. We further restrict our analysis to equatorial orbits with eccentricity $e = 0.5$ and latus rectum on the order of $l \sim r_{\text{ISCO}}$. In that case, $L = 0.53 \times 10^7 \text{ cm}^2/\text{sec}$, $\Omega \equiv \frac{g\mu}{mL} \simeq 0.81 \times 10^5 \text{ cm}$, $\mu = r_S/2l = 1/6$, and $\frac{\Omega}{c} \simeq 1.41 \times 10^{-3}$ is indeed less than 1, so our approximations are valid. We obtain an estimate of the ratio $L/El^2 \simeq \pm 0.79 \times 10^7 \text{ sec/cm}^2$, and $Nl/c \simeq 0.141$. The total radiated power \mathcal{P} is then obtained by integrating Eq. (50) over all frequencies and solid angles, and by dividing with T , namely $\mathcal{P} \simeq 1.4 \times 10^{-23} \text{ erg/sec}$. This yields a timescale for orbit evolution on the order of $\tau \sim m_e v^2 / \mathcal{P} \sim 10^{16} \text{ sec}$, which is dynamically unimportant. In a future publication, we will relax our current working assumption that $\frac{\Omega}{c} \ll 1$, and will also consider higher values of the magnetic field for which radiation effects will become astrophysically significant.

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Appendix A

The explicit expressions for X, Y, Z are

$$X \equiv (\epsilon_{123})^2 [g^{33}n^2(n^1v^2 - n^2v^1) - g^{22}n^3(n^3v^1 - n^1v^3)], \quad (57)$$

$$Y \equiv (\epsilon_{123})^2 [-g^{33}n^1(n^1v^2 - n^2v^1) + g^{11}n^3(n^2v^3 - n^3v^2)], \quad (58)$$

$$Z \equiv (\epsilon_{123})^2 [g^{22}n^1(-n^1v^3 + n^3v^1) - g^{11}n^2(n^2v^3 - n^3v^2)], \quad (59)$$

where the components n^i, v^i are computed in the curved space time and $\epsilon_{123} = \frac{r^2 \sin \theta}{\sqrt{(1-r_S/r)}}$. Using (34), (35) and their derivatives, we find:

$$\begin{aligned}
X_0 = 2lN \sin(2Nt) & \left\{ \cos^2 \lambda e(1-e^2) \times [1-2\mu(1+e^2) - (1-4\mu)e \cos(2Nt)]^{1/2} \right. \\
& \times [1-e \cos(2Nt)]^{-1/2} [(1+e^2) - 2e \cos(2Nt)]^{-2} \\
& - \sin^2 \lambda (1-e^2) [\cos(2Nt) - e] \\
& \times [1-2\mu(1+e^2) - (1-4\mu)e \cos(2Nt)]^{1/2} \\
& \left. \times [1-e \cos(2Nt)]^{-3/2} [(1+e^2) - 2e \cos(2Nt)]^{-2} \right\}, \quad (60)
\end{aligned}$$

$$\frac{1}{r} Y_0 = -2lN \sin \lambda \cos \lambda \frac{\sqrt{(1-e^2)}[(1+e^2) \cos(2Nt) - 2e]}{[(1+e^2) - 2e \cos(2Nt)]^2}, \quad (61)$$

$$\begin{aligned}
\frac{1}{r} Z_0 = -2lN & \left\{ \sin^2 \lambda \frac{(1-e^2)^{3/2} \sin^2(2Nt)}{[1-e \cos(2Nt)][(1+e^2) - 2e \cos(2Nt)]^2} \right. \\
& \left. + \cos^2 \lambda \frac{\sqrt{1-e^2}}{[(1+e^2) - 2e \cos(2Nt)]} \right\}, \quad (62)
\end{aligned}$$

while the term $\frac{r \sin \lambda \sin \phi}{\sqrt{(1-r_S/r)}}$ in (40) becomes

$$\begin{aligned}
\frac{r \sin \lambda \sin \phi}{\sqrt{(1-r_S/r)}} &= l \sqrt{1-e^2} \sin \lambda \sin(2Nt) \times [1-e \cos(2Nt)]^{-3/2} \\
&\times [1-2\mu(1+e^2) - (1-4\mu)e \cos(2Nt)]^{-1/2}. \quad (63)
\end{aligned}$$

Keeping terms up to e^2 , an approximate form of the (60-63) is

$$X_0 = 2lN \sin(2Nt) \cos^2 \lambda [(1-2\mu)^{1/4} e + O(e^2)], \quad (64)$$

$$\frac{1}{r} Y_0 = -2lN \sin \lambda \cos \lambda \times \{\cos(2Nt) + 2[\cos^2(2Nt) - 1]e + O(e^2)\}, \quad (65)$$

$$\frac{1}{r} Z_0 = -2lN \{\sin^2 \lambda \sin^2(2Nt) \times [1 + 5 \cos(2Nt)e + O(e^2)]\}, \quad (66)$$

and

$$\mathbf{n} \cdot \mathbf{r} = l \sin \lambda \sin(2Nt) \left[\frac{1}{\sqrt{1-2\mu}} + \frac{(2-5\mu)}{(1-2\mu)^{3/2}} \cos(2Nt)e + O(e^2) \right]. \quad (67)$$

Appendix B

In (31) we computed the ratio L/E with the aid of (13), (14). In this case, we find

$$\frac{E^2}{L^2} = \frac{2}{r_{sl}} \left[1 - (3 + e^2) \left(\mu - \frac{\Omega}{l} \right) \right] - \frac{1}{l^2} \left[1 - 4 \left(\mu - \frac{\Omega}{l} \right) (1 - e^2) \right]. \quad (68)$$

Furthermore, (68) gives

$$\frac{L}{E} = \pm \sqrt{Ml} \left\{ (1 - 2\mu)^2 - 4\mu^2 e^2 + \frac{\Omega}{l} [(3 + e^2) - 4\mu(1 - e^2)] \right\}^{-1/2}. \quad (69)$$

In (69), setting $\Omega = 0$ and $e = 0$, we recover Breuer's result [24]. In the case where $\Omega = 0$ and $e \neq 0$, we recover the result obtained in [16].

From (31) and (68) we find

$$2N = \pm \frac{M^{1/2}}{l^{3/2}} \left\{ (1 - 2\mu)^2 - 4\mu^2 e^2 + \frac{\Omega}{l} [(3 + e^2) - 4\mu(1 - e^2)] \right\}^{-1/2} \\ \times \left[1 + \frac{\Omega}{l} (1 - 2e) \right]^{-1} \times \left[1 - (3 - e) \left(2\mu - 2\frac{\Omega}{l} \right) \right]^{1/2}. \quad (70)$$

In the limit $\Omega = 0$, we recover the coefficient of the Eq. (136) in [16]. Observe, that on the right side of (70) the coefficient $\pm M^{1/2}/l^{3/2}$ corresponds to the Newtonian frequency of a Keplerian orbit at $r = l$ in $c = G = 1$ units.

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